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LETTER TO THE EDITOR

Classification and universal properties of Sierpinski carpets

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Abstract. The type of critical points for the Potts model is found to be dependent on the structure of Sierpinski carpets. All possible types of structure for Sierpinski carpets with two interactions after bond moving are found. We present general Sierpinski carpets and new parameters are proposed and used to describe them. The lacunarity expression is revised. We discuss and propose a universal classification. Critical points, eigenvalues and flow diagrams are presented. Some fixed points display negative eigenvalues.

The Sierpinski carpets are two-dimensional fractals with infinite ramification (Mandelbrot 1977, 1982). Phase transition at finite temperature is possible in these systems. The Ising and Potts models on Sierpinski carpets have been studied and some fixed points and critical exponents have been obtained (Gefen *et al* 1984, Lin 1986). However, so far only one type of Sierpinski carpets has been studied—carpets with central cutout.

We have pointed out that on the central cutout Sierpinski carpets marginal critical points will appear (Lin 1986). In that paper we also conjectured that the marginal critical points (K, ∞) and (∞, K_w) are due to the peculiarity of eliminations. In this letter we will study this conjecture. We will give all possible types of elimination (structure) with two interactions after bond moving and show that different types of elimination of Sierpinski carpets correspond to different types of fixed points for Potts model. There are structures in which marginal critical exponents disappear.

First we introduce the extension of Sierpinski carpets. We call the Sierpinski carpets described by Mandelbrot and Gefen *et al* (Mandelbrot 1977, 1982, Gefen *et al* 1980, 1984) regular carpets, i.e. boundary subsquares are retained and eliminated subsquares are arranged in a square array (e.g. figure 4). For these carpets parameters (b, l) can be used to describe the structures (Gefen *et al* 1980, 1984). We now introduce general Sierpinski carpets. Figure 1 is an example. We see that the elimination manner of figure 1 differs from that of a regular one—some of the boundary subsquares are cut out and the eliminated subsquares are no longer arranged in a square array (but still in a symmetrical manner). It is obvious that (b, l) cannot serve the general carpet like figure 1. We suggest the parameters (b, p, t) for the description of a general carpet, where p is the number of eliminated subsquares from a square containing b subsquares and t is the number of rows (or columns) in which boundary subsquares are cut out (if boundary subsquares are cut out) or there are eliminated subsquares (if boundary subsquares are not cut out, see figures 1 and 3). Then the formula for the fractal dimension D of Sierpinski carpets should be, in these new parameters,

$$D = \ln(b^2 - p)/\ln b. \tag{1}$$

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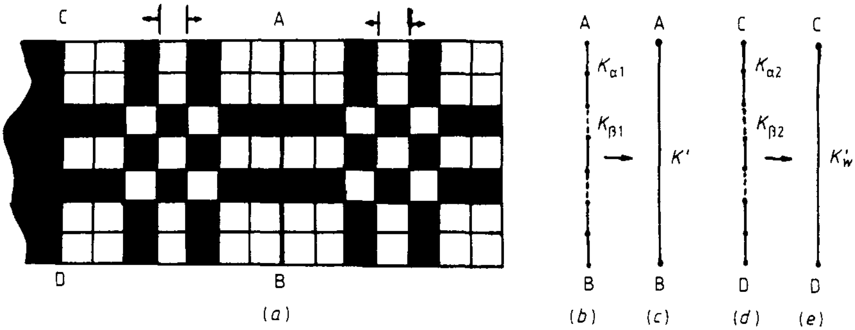


Figure 1. A general Sierpinski carpet with two interactions after bond moving, type $(C_3, C_7) = 0$, $b = 7$, $p = 20$ and $t = 2$: (a) the unrenormalised carpet, and (b) bonds between renormalised sites A and B after bond moving. There are only two kinds of bond (interaction) $K_{\alpha 1} = 3K + 4K_w$ and $K_{\beta 1} = 4K_w$, (c) renormalised bond between A and B, K', (d) results of bond moving for bonds between C and D. $K_{\alpha 2} = K + 3K_w$ and $K_{\beta 2} = 2K_w$; (e) renormalised bond K'_w.

We apply bond-moving renormalisation to analyse the critical behaviour of the Potts model on Sierpinski carpets. Two kinds of interactions K and K_w were defined on a carpet (Gefen *et al* 1984). We here consider only that kind of structure from which there are only two interactions between any two renormalised sites after bond moving (figures 1 and 2). With our new parameters we can easily write the approximate recursion relation and four matrix elements determining eigenvalues for the q-state Potts model on general Sierpinski carpets, which are similar to the relations we have obtained for regular carpets (Lin 1986):

$$e^{-K'_i} = \frac{[1+(q-1)\,e^{-K_{\alpha i}}]^{b-t}[1+(q-1)\,e^{-K_{\beta i}}]^t - (1-e^{-K_{\alpha i}})^{b-t}(1-e^{-K_{\beta i}})^t}{[1+(q-1)\,e^{-K_{\alpha i}}]^{b-t}[1+(q-1)\,e^{-K_{\beta i}}]^t + (q-1)(1-e^{-K_{\alpha i}})^{b-t}(1-e^{-K_{\beta i}})^t} \qquad i = 1,2 \tag{2}$$

$$\left(\frac{\partial K'_i}{\partial K_j}\right) = \frac{q^2 G_i}{(1-G_i)[1+(q-1)G_i]}\left[\frac{(b-t)\,e^{-K_{\alpha i}}}{F_{1i}F_{2i}}\left(\frac{\partial K_{\alpha i}}{\partial K_j}\right) + \frac{t\,e^{-K_{\beta i}}}{F_{3i}F_{4i}}\left(\frac{\partial K_{\beta i}}{\partial K_j}\right)\right] \qquad i,j = 1,2 \tag{3}$$

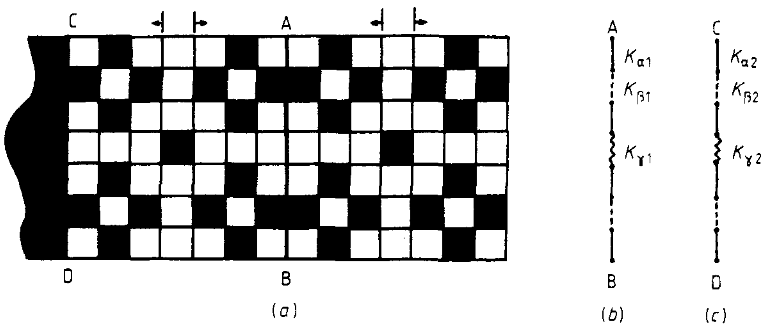


Figure 2. A Sierpinski carpet with three interactions after bond moving (see the caption of figure 1). It is easy to see that $K_{\alpha 1} = 3K + 4K_w$, $K_{\beta 1} = 6K_w$, $K_{\gamma 1} = 5K + 2K_w$; $K_{\alpha 2} = K + 3K_w$, $K_{\beta 2} = 3K_w$ and $K_{\gamma 2} = 2K + 2K_w$. This kind of structure is not included in our discussion.

where $K_1' = K'$ and $K_2' = K_w'$ are renormalised interactions (figure 1). $K_{\alpha i}$ and $K_{\beta i}$ are effective interactions after bond moving (before decimation). $K_{\alpha 1}$ and $K_{\beta 1}$ are to be renormalised to K' and $K_{\alpha 2}$ and $K_{\beta 2}$ to K_w' (see figure 1). In (3) we have

$$F_{1i} = [1+(q-1)\,e^{-K_{\alpha i}}] \qquad F_{2i} = (1-e^{-K_{\alpha i}})$$
$$F_{3i} = [1+(q-1)\,e^{-K_{\beta i}}] \qquad F_{4i} = (1-e^{-K_{\beta i}})$$
$$G_i = (F_{2i}/F_{1i})^{b-t}(F_{4i}/F_{3i})^t.$$

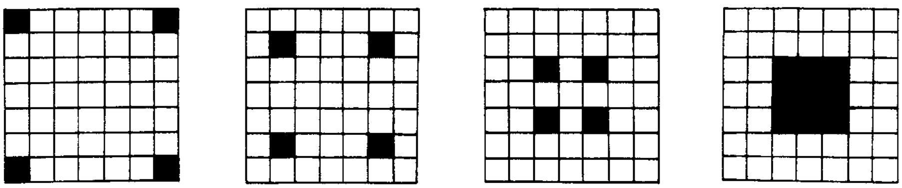
Different types of structure produce different $K_{\alpha i}$ and $K_{\beta i}$ and therefore different types of fixed point. The most general form of $K_{\alpha i}$ and $K_{\beta i}$ is

$$K_{\alpha 1} = C_1 K + C_2 K_w \qquad K_{\beta 1} = C_3 K + C_4 K_w$$
$$K_{\alpha 2} = C_5 K + C_6 K_w \qquad K_{\beta 2} = C_7 K + C_8 K_w. \tag{4}$$

Combinations of zero coefficients (C_1-C_8) lead to various types of structure. We find that there are, altogether, seven combinations (i.e. seven types of structure). They are

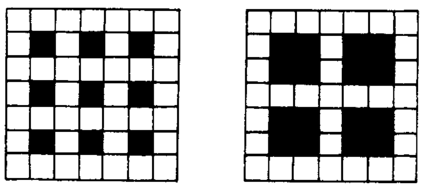
$$\varnothing,\ C_2=0,\ C_5=0,\ (C_2, C_7)=0,\ (C_3, C_7)=0,\ (C_2, C_3, C_7)=0,\ (C_3, C_5, C_7)=0. \tag{5}$$

Here $\varnothing$ stands for the case where none of the eight coefficients is zero. $(C_i, C_j)=0$ means $C_i=0$ and $C_j=0$ simultaneously, and the same notation applies for $(C_i, C_j, C_k)=0$. We point out that when $b<6$ some of the seven types (5) will disappear. For example, type $\varnothing$ disappears and type $(C_3, C_1)=0$ remains with only the cross cutout structure (no fixed point, figure 6) when $b=5$, and in the case of $b=3$, there remain only types $(C_2, C_7)=0$, $(C_2, C_3, C_7)=0$ and $(C_3, C_5, C_7)=0$. Within a type of structure there are several distinct eliminations. In figure 1 and figures 3-9 we present some typical structures. We summarise the lacunarities, fixed points and eigenvalues for each of the seven types (5) in table 1. Flow diagrams are shown in figure 10. For convenience of drawing we draw the flow diagrams in (e^{-K}, e^{-K_w}) space.



(*a*) (*b*) (*c*) (*d*)

Figure 3. Sierpinski carpets, type $C_2=0$ in classification (5), $b=7$: (*a*) $p=4$, $t=2$; (*b*), (*c*) $p=4$, $t=2$. They have the same symmetry (lacunarities are different) and the same fractal dimension D, so the critical behaviours are the same. The symmetry differs from that of (*a*). (*d*) $p=9$, $t=3$, having different symmetry from figure 4(*a*), although the eliminated subsquares for both are identical.



(*a*) (*b*)

Figure 4. Regular carpets, type $(C_2, C_7)=0$, $b=7$: (*a*) $p=9$, $t=3$; (*b*) $p=16$, $t=4$.

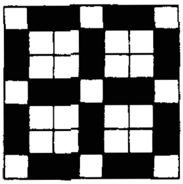


Figure 5. A general carpet, type $C_5 = 0$, $b = 7$, $p = 24$, $t = 4$.

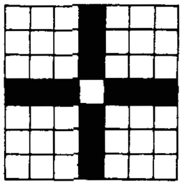
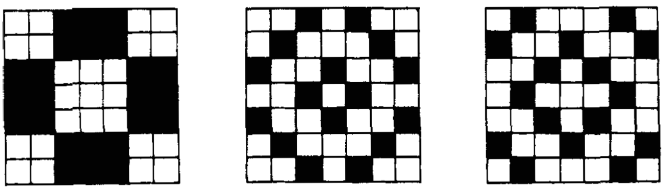
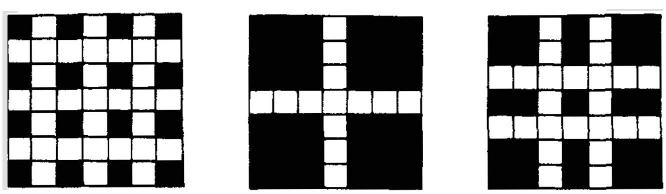


Figure 6. A general carpet, 'cross cutout' carpet, $b = 7$, $p = 12$, $t = 1$, type $(C_3, C_7) = 0$, belonging to the same type as figure 1 but having no fixed point.



(*a*) (*b*) (*c*)

Figure 7. General carpets, type $\varnothing$, $b = 7$: (*a*) $p = 24$, $t = 3$; (*b*), (*c*) possessing the same fractal dimension D and symmetry (lacunarities are different) and the same universal class.



(*a*) (*b*) (*c*)

Figure 8. General carpets, type $(C_2, C_3, C_7) = 0$, $b = 7$: (*a*) $p = 16$, $t = 4$; (*b*) 'cross remaining' carpet, $p = 36$, $t = 6$. Unlike (*a*), (*b*) has no fixed point. $L = 0$ by expression (6) while (7) gives $L = 0.4936$. (*c*) $p = 25$, $t = 5$; another example of improper zero lacunarity for inhomogeneous carpet calculated from expression (6), $L = 0$; (7) gives $L = 0.2849$.

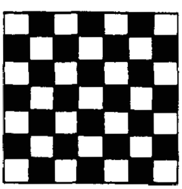


Figure 9. A general carpet, type $(C_3, C_5, C_7) = 0$, $b = 7$, $p = 24$, $t = 3$. There is only one kind of interaction in this type of structure, K_w.

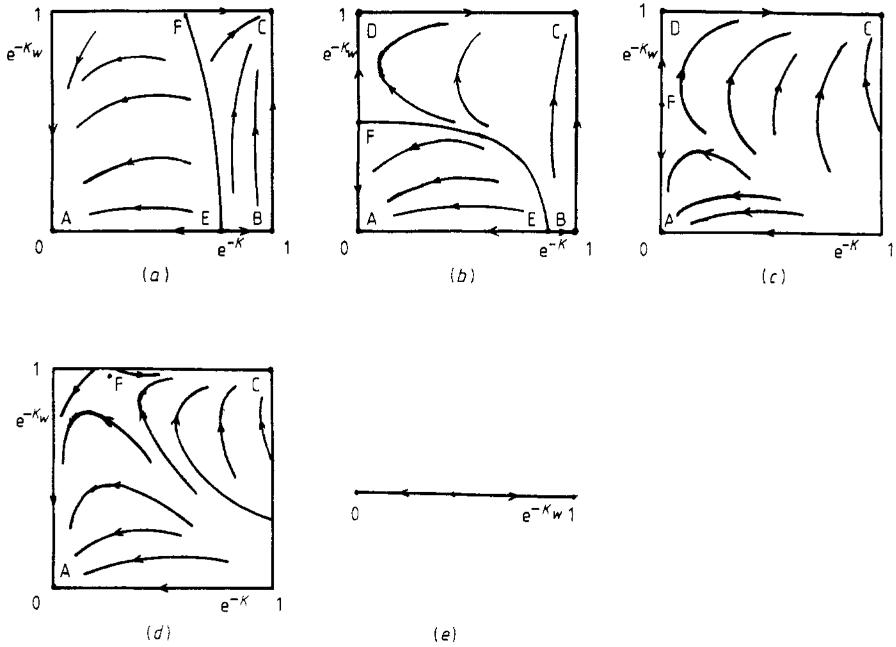


Figure 10. Flow diagrams for the Potts model on general Sierpinski carpets. E and F are critical points, A, B, C and D are trivial fixed points (see table 1 for details). (*a*) Type $C_2=0$, figure 3(*b*); (*b*) type $(C_2, C_7)=0$, figure 4(*b*); (*c*) type $C_5=0$, figure 5; (*d*) type $\varnothing$, figure 7(*a*); (*e*) type $(C_3, C_5, C_7)=0$, figure 9, single-interaction structure.

In table 1 we list the eigenvalues λ_i instead of the critical exponents y_i because some fixed points display negative eigenvalues. We know $y_i=\ln\lambda_i/\ln b$, b here playing the role of the scale factor of the RG. For positive λ_i, y_i is easy to obtain, but y_i can not be determined for negative λ_i. One can easily see that the flow lines for the structures with negative eigenvalues intersect. This is not permitted for a flow diagram. So we exclude these structures from phase transition systems (types $(C_3, C_7)=0$ and $(C_2, C_3, C_7)=0$, see table 1). Although critical exponents y_i for these structures can be formally calculated ($|\lambda_i|<1$ for every negative λ_i and there is always a relevant eigenvalue $\lambda_j>1$ accompanying each negative λ_i) we do not present the flow diagrams for the structures with negative eigenvalues.

There are two special cases in table 1: the 'cross cutout' carpet and 'cross remaining' carpet (figures 6 and 8(*b*) respectively). They belong to the types $(C_3, C_7)=0$ and $(C_2, C_3, C_7)=0$ respectively, but do not behave like the other structures in their respective types—there is no fixed point in these two structures (except for trivial points). This particular property of these cross carpets is independent of b and q values, i.e. the cross carpets have no fixed point in any b or q value. Among seven types of structures (5), type $(C_3, C_5, C_7)=0$ is a single-interaction structure, where K_w only exists (figure 9). One can easily write $K_{\alpha i}$ and $K_{\beta i}$ for figure 9: $K_{\beta 1}=4K_w$ and $K_{\beta 2}=3K_w$. So this type is equivalent to $(C_1, C_2, C_3, C_5, C_6, C_7)=0$.

Lacunarities presented in table 1 are calculated by a formula slightly different from that in our previous paper (Lin and Yang 1986), where the lacunarity L is defined by

$$L=\frac{1}{N}\sum_{s=l}^{b} L(s). \tag{6}$$

Table 1. Results for the Potts model on various Sierpinski carpets. In this table we list for the three-state ($q = 3$) Potts model only general carpets with $b = 7$. However, the classification (5) is valid for any value $b > 6$ and with the general expressions (2) and (3) one can easily obtain results for the q-state Potts model on general Sierpinski carpets with any b value.

						Point E		Point F†							
Type	Figure	p	t	D	L	e^{-K^*}	λ_K	e^{-K^*}	$e^{-K_n^*}$	λ_1	λ_2	$e^{-K_n^*}$	λ_{K_n}	(0, 1)	(1, 0)‡
C_2	3(*a*)	4	2	1.956	0.0780	0.779	3.970	0.666	0.995	3.912	0.013				×
	3(*b*)	4	2	1.956	0.0891	0.779	3.970	0.614	0.996	3.302	0.024				×
	3(*c*)	4	2	1.956	0.1016	0.779	3.970	0.614	0.996	3.302	0.024				×
	3(*d*)	9	3	1.896	0.2411	0.815	3.550	0.545	0.995	2.980	0.035				×
$C_2\,C_7$	4(*a*)	9	3	1.896	0.1161	0.815	3.550					0.711	2.685	×	×
	4(*b*)	16	4	1.797	0.2421	0.866	2.954					0.492	2.685	×	×
C_5	5	24	4	1.654	0.2418							0.580	2.453	×	
∅	7(*a*)	24	3	1.654	0.3372			0.250	0.972	2.417	0.082				
	7(*b*)	16	2	1.797	0.1639			0.289	0.986	2.574	0.073				
	7(*c*)	16	2	1.797	0.1627			0.289	0.986	2.574	0.073				
$C_3\,C_5\,C_7$	9	24	3	1.654	0.1883				0.445		7.715				
$C_3\,C_7$	1	20	2	1.730	0.2191			0.082	0.448	2.098	−0.162				
	6	12	1	1.856	0.1886			No fixed point							
$C_2\,C_3\,C_7$	8(*a*)	16	4	1.797	0.1621	0.866	2.954	0.058	0.492	2.688	−0.004				×
	8(*b*)	36	6	1.318	0.5183			No fixed point							×
	8(*c*)	25	5	1.633	0.2992	0.946	2.066	0.006	0.190	2.057	-0.38×10^{-5}				×

† For point E, the fixed point is of $(e^{-K^*}, 0)$ type and the eigenvalue of $(\lambda_K, 1)$ type. Similarly, for point F, fixed point $e^{-K_n^*}$ means $(0, e^{-K_n^*})$ and λ_{K_n} means $(1, \lambda_{K_n})$.
‡ Each structure has trivial fixed points (0, 0) and (1, 1) (points A and C in figure 10).

This works well for regular carpets. Now for a general carpet there are some problems with using (6): the parameter l is not a valid parameter and, more importantly, there are inhomogeneous carpets on which (6) would produce zero lacunarity (figures 8(b) and 8(c)). To calculate the lacunarity of a general carpet we amend expression (6) and use an all-scale average of $L(s)$ to define L

$$L=\frac{1}{b-1}\sum_{s=1}^{b-1} L(s) \tag{7}$$

where $L(s)$ is the same as in (6). Expression (7) removes the improper zero lacunarities for inhomogeneous carpets produced by (6). Having analysed numerical results of $L(s)$ and the new L, we find that the discreteness of elimination has little effect on $L(s)$ with small S and therefore little effect on L. As a whole, $L(s)$ decreases as S increases for all S values 1 to b-1 (see Lin and Yang 1986). In view of the facts above we suggest that expression (6) be replaced by (7). (7) has been used in table 1. Lacunarities calculated by the new expression (7) agree with the direct observations.

Let us compare the lacunarities of figures 5, 7(a) and 9 (they have the same parameters b and p). We can distinguish the homogeneities among them intuitively: figure 9 is the most homogeneous and figure 5 the second; (7) tells us that their lacunarities are 0.1883, 0.2418 and 0.3373 respectively. Figures 8(a), 7(c), 7(b) and 4(b) are another example (see table 1).

We argue that L cannot serve as a classifying parameter, because there are carpets having different lacunarities but possessing the same fixed points and critical exponents (figures 3(b) and 3(c), 7(b) and 7(c)). We propose that, besides the fractal dimension D, a symmetrical factor be a classifying parameter. In other words, we think that when two carpets have the same symmetry they would have the same critical exponents if they possessed an identical fractal dimension D. If the eliminated blocks of a structure are moved symmetrically to form another carpet structure, then the two structures have the same symmetry, fixed points and exponents (e.g. figures 3(b) and 3(c); figures 7(b) and 7(c), see table 1). In the symmetrical moving of eliminated blocks, the boundary subsquares are particular: the boundary eliminated subsquares cannot be moved away from boundary subsquares and vice versa. If not, the symmetry is considered changed. Figure 7(c) preserves the symmetry oı figure 7(b), but figure 8(a) changes the symmetry (it moves two additional eliminated subsquares to the boundary of each side). So the critical behaviours of figure 8(a) differ from that of figures 7(b) and 7(c) (see table 1; another example is figures 3(b), 3(c) and 3(a)). It should be noted that the eliminated subsquares should be moved in blocks, otherwise the symmetry cannot be preserved. Figures 3(d) and 4(a) have the same eliminated subsquares but different symmetry. However, all fixed points of $(e^{-K^*}, 0)$ type (and corresponding eigenvalues $(\lambda_K, 1)$) with the same fractal dimension D are identical, no matter what lacunarity and symmetry they have (see table 1, and compare figure 4(b) with 8(a), 3(d) with 4(a) and 3(a) with 3(b)). We would like to mention that our universality discussion is based on the approximate bond-moving renormalisation. We look forward to hearing more results on this topic.

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